§ 8 THE HOLOMORPHIC CASE

In this section M is supposed to be a complex manifold. For a complex line bundle $\pi : L \to M$ the following additional structures are to be considered:

1) Connections ∇ on L,
2) Hermitian metric H on L,
3) Holomorphic structure on L.

M is a complex manifold of (complex) dimension n if M is a smooth manifold of real dimension $2n$ and one has an atlas of holomorphic (or complex) charts

$$\xi_j : U_j \to V_j \subset \mathbb{C}^n, \quad j \in I,$$

where $(U_j)_{j\in I}$ is an open cover of M, $V_j \subset \mathbb{C}^n$ is open, ξ_j is smooth, and the maps

$$\xi_k \circ \xi_j^{-1} : \xi_j(U_j \cap U_k) \longrightarrow \xi_k(U_j \cap U_k)$$

are biholomorphic, i.e. $\xi_k \circ \xi_j^{-1}$ and $\xi_j \circ \xi_k^{-1}$ are holomorphic.

A map $\varphi : V \to V'$ between open subsets $V \subset \mathbb{C}^n$ and $V' \subset \mathbb{C}^m$ is holomorphic, if for every $a \in V$ and $z \in \mathbb{C}^n$, the map

$$\lambda \longmapsto \varphi(a + \lambda z)$$

is a holomorphic map in one variable.

(8.1) DEFINITION: A complex line bundle $\pi : L \to M$ over a complex manifold M is HOLOMORPHIC if L is a complex manifold, $\pi : L \to M$ is a holomorphic map and there exists an open cover $(U_j)_{j\in I}$ of M with trivializations

$$\varphi_j : L_{U_j} \longrightarrow U_j \times \mathbb{C}$$

which are holomorphic maps.

Similar to our results on general complex line bundles over a manifold the holomorphic line bundles are given by transition functions $(g_{jk})_{j,k\in I}$, but now, they are all holomorphic functions

$$g_{jk} : U_{jk} \to \mathbb{C}^\times \ , \quad \text{or} \quad g_{jk} \in \mathcal{O}^\times(U_{jk}).$$

The group of isomorphism classes of holomorphic line bundles over the complex manifold M is

$$H^1(M, \mathcal{O}^\times).$$

A section $s : U \to L$ of a holomorphic line bundle L over an open subset $U \subset M$ is called HOLOMORPHIC if s is a holomorphic map. $\Gamma_{hol}(U,L) \subset \Gamma(U,L)$ denotes the subspace of holomorphic sections

$$\Gamma_{hol}(U,L) := \{ s \in \Gamma(U,L) \mid s \text{ holomorphic} \}$$

$\Gamma_{hol}(U,L)$ is a complex vector space and a module over the ring $\mathcal{O}(U)$ of holomorphic functions on $U \subset M$.

(8.2) <u>DEFINITION</u>: Let $L \xrightarrow{\pi} M$ be a holomorphic line bundle with a connection ∇ on L.

1° ∇ is a HOLOMORPHIC CONNECTION if for any local holomorphic section $s \in \Gamma(U, L)$ the map

$$X \mapsto \nabla_X s \ , \quad X \in \mathcal{D}(U) \text{ and } X \text{ holomorphic}$$

is a holomorphic one form. I.e. in local holomorphic coordinates $\xi = (z^1, \dots z^n) : U \to V \subset \mathbb{C}^n$

$$\nabla_X s = f_j \, dz^j$$

with holomorphic $f_j : U \to \mathbb{C}$.

2° ∇ is COMPATIBLE with the holomorphic structure on L if for any local holomorphic section $s \in \Gamma_{hol}(U, L)$, the one form

$$X \mapsto \nabla_X s$$

is of pure type $(1,0)$, i.e. in local holom. coordinates

$$\nabla_X s = f_j \, dz^j \ , \quad f_j \in \mathcal{E}(U) .$$

(8.3) <u>PROPOSITION</u>: Every holomorphic line bundle L over the complex manifold M admits a connection ∇ compatible with the holomorphic structure on the line bundle L.

(8.4) PROPOSITION: Let L be a holomorphic line bundle and let ∇ be a connection on L which is compatible with the holomorphic structure on L. Then the curvature $\mathrm{Curv}(\nabla, L) = \Omega$ has components only of type $(2,0)$ and $(1,1)$, i.e in local coordinates $\xi = (z^1, \dots z^n) : U \to V \subset \mathbb{C}^n$:

$$\Omega = \omega_{jk}\, dz^j \wedge dz^k + \rho_{jk}\, dz^j \wedge d\bar{z}^k \quad , \; \omega_{jk}, \rho_{jk} \in \mathcal{E}(U).$$

(8.5) PROPOSITION: Let $L \to M$ be a holomorphic line bundle which is also endowed with a Hermitian metric $H : L \times L \to \mathbb{C}$. Then there exists a unique connection ∇ which is compatible both with the holomorphic structure and the Hermitian structure. The curvature is of type $(1,1)$.

As an example the tautological line bundle

$$T = H(-1) \longrightarrow \mathbb{P}_n(\mathbb{C})$$

is a holomorphic line bundle. A natural connection is

$$\nabla_X s = \frac{\bar{z}^j X_j}{\sum |z^j|^2} (z_1, z_2, \dots z_n)$$

∇ is compatible with the complex structure and also with the Hermitian metric

$$H(z, z') = \sum_{j=1}^{n} \bar{z}^j z'^j$$

(8.6) <u>PROPOSITION</u>: Let $L \to M$ be a smooth complex line bundle over the complex manifold M equipped with a connection whose curvature is purely of type $(1,1)$. Then there exists a unique holomorphic structure on L for which a local section s of L is holomorphic if and only if $X \mapsto \nabla_X s$ is a one form of type $(1,0)$.

Proofs can be found in Brylinski, for example.